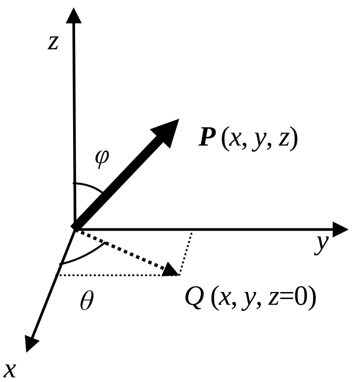
直交座標系(x, y, z)から極座標系(ρ, θ, ψ)への変換

直交座標系での支配方程式は以下である。

$$K_x \frac{\partial^2 s}{\partial x^2} + K_y \frac{\partial^2 s}{\partial y^2} + K_z \frac{\partial^2 s}{\partial z^2} = Ss \frac{\partial s}{\partial t} \tag{1}$$

ここで，曲座標系での標記を以下と定義する。

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi), \quad \rho^2 = x^2 + y^2 + z^2$$
$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \varphi = cos^{-1}\left(\frac{z}{\rho}\right), \quad \theta = cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \tag{2}$$



透水係数は等方性とし，$K=K_x=K_y=K_z$

結果として，以下の式となる。

$$\frac{\partial^2 s}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial s}{\partial r} + \frac{1}{\rho^2}\frac{\partial^2 s}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial s}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 s}{\partial \theta^2} = \frac{Ss}{K}\frac{\partial s}{\partial t} \tag{3}$$

以下に，この結果に至る誘導を試みる。

曲座標系成分を直交座標系成分で微分したものは以下となる。

$$\frac{\partial \rho}{\partial x} = \sin\varphi\cos\theta, \quad \frac{\partial \rho}{\partial y} = \sin\varphi\sin\theta, \quad \frac{\partial \rho}{\partial z} = \cos\varphi$$
$$\frac{\partial \varphi}{\partial x} = \frac{1}{\rho}\cos\varphi\cos\theta, \quad \frac{\partial \varphi}{\partial y} = \frac{1}{\rho}\cos\varphi\sin\theta, \quad \frac{\partial \varphi}{\partial z} = \frac{-1}{\rho}\sin\varphi \tag{4}$$
$$\frac{\partial \theta}{\partial x} = \frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}, \quad \frac{\partial \theta}{\partial z} = 0$$

また，x についての微分は以下のように定義される。

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} \tag{5}$$

よって，これより以下を誘導する。

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\right)$$
$$= \left(\frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\right)\left(\frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\right) \tag{6}$$

この式を展開して，以下の 3 パートに分割して整理する。

$$①:\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)=\frac{\partial\rho}{\partial x}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\rho}\left(\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\rho}\left(\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)\right]$$

$$=\frac{\partial\rho}{\partial x}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\rho}{\partial x}\right)\frac{\partial}{\partial\rho}+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\rho}\left(\frac{\partial\theta}{\partial x}\right)\frac{\partial}{\partial\theta}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\rho}\left(\frac{\partial\varphi}{\partial x}\right)\frac{\partial}{\partial\varphi}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\varphi}\right)\right] \tag{7}$$

$$②:\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)=\frac{\partial\theta}{\partial x}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)\right]$$
$$=\frac{\partial\theta}{\partial x}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\rho}{\partial x}\right)\frac{\partial}{\partial\rho}+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\theta}{\partial x}\right)\frac{\partial}{\partial\theta}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial x}\right)\frac{\partial}{\partial\varphi}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\varphi}\right)\right] \tag{8}$$

$$③:\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)=\frac{\partial\varphi}{\partial x}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)\right]$$
$$=\frac{\partial\varphi}{\partial x}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial\rho}{\partial x}\right)\frac{\partial}{\partial\rho}+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\rho}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\partial\theta}{\partial x}\right)\frac{\partial}{\partial\theta}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\partial\varphi}{\partial x}\right)\frac{\partial}{\partial\varphi}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right)\right] \tag{9}$$

同様に，y, z 座標成分の微分($\partial^2/\partial y^2$，$\partial^2/\partial z^2$)を行い，上記とあわせ，曲座標系成分項による微分項毎に係数を集約する。

$$\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\rho}\right):$$

$$\frac{\partial\rho}{\partial x}\left[\frac{\partial\rho}{\partial x}\right]+\frac{\partial\rho}{\partial y}\left[\frac{\partial\rho}{\partial y}\right]+\frac{\partial\rho}{\partial z}\left[\frac{\partial\rho}{\partial z}\right]$$
$$=\sin^2\varphi\cos^2\theta+\sin^2\varphi\sin^2\theta+\cos^2\varphi=\sin^2\varphi\left(\cos^2\theta+\sin^2\theta\right)+\cos^2\varphi \tag{10}$$
$$=\sin^2\varphi+\cos^2\varphi=1$$

$$\frac{\partial}{\partial\rho}:$$

$$\frac{\partial\rho}{\partial x}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\rho}{\partial x}\right)\right]+\frac{\partial\theta}{\partial x}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\rho}{\partial x}\right)\right]+\frac{\partial\varphi}{\partial x}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial\rho}{\partial x}\right)\right]$$
$$+\frac{\partial\rho}{\partial y}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\rho}{\partial y}\right)\right]+\frac{\partial\theta}{\partial y}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\rho}{\partial y}\right)\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial\rho}{\partial y}\right)\right]$$
$$+\frac{\partial\rho}{\partial z}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\rho}{\partial z}\right)\right]+\frac{\partial\theta}{\partial z}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\rho}{\partial z}\right)\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\partial}{\partial\varphi}\left(\frac{\partial\rho}{\partial z}\right)\right]$$
$$=\frac{\partial\rho}{\partial x}\left[\frac{\partial}{\partial\rho}(\sin\varphi\cos\theta)\right]+\frac{\partial\theta}{\partial x}\left[\frac{\partial}{\partial\theta}(\sin\varphi\cos\theta)\right]+\frac{\partial\varphi}{\partial x}\left[\frac{\partial}{\partial\varphi}(\sin\varphi\cos\theta)\right]$$
$$+\frac{\partial\rho}{\partial y}\left[\frac{\partial}{\partial\rho}(\sin\varphi\sin\theta)\right]+\frac{\partial\theta}{\partial y}\left[\frac{\partial}{\partial\theta}(\sin\varphi\sin\theta)\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\partial}{\partial\varphi}(\sin\varphi\sin\theta)\right]$$
$$+\frac{\partial\rho}{\partial z}\left[\frac{\partial}{\partial\rho}(\cos\varphi)\right]+\frac{\partial\theta}{\partial z}\left[\frac{\partial}{\partial\theta}(\cos\varphi)\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\partial}{\partial\varphi}(\cos\varphi)\right]$$

$$=\frac{\partial\rho}{\partial x}[0]+\frac{-\sin\theta}{\rho\sin\varphi}[-\sin\varphi\sin\theta]+\frac{\cos\varphi\cos\theta}{\rho}[\cos\varphi\cos\theta]$$
$$+\frac{\partial\rho}{\partial y}[0]+\frac{\cos\theta}{\rho\sin\varphi}[\sin\varphi\cos\theta]+\frac{\cos\varphi\sin\theta}{\rho}[\cos\varphi\sin\theta]$$
$$+\frac{\partial\rho}{\partial z}[0]+\frac{d\theta}{dz}[0]+\frac{-\sin\varphi}{\rho}[-\sin\varphi]$$

$$
\begin{aligned}
&= \frac{\sin^2\theta}{\rho} + \frac{\cos^2\varphi\cos^2\theta}{\rho} + \frac{\cos^2\theta}{\rho} + \frac{\cos^2\varphi\sin^2\theta}{\rho} + \frac{\sin^2\varphi}{\rho} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\rho} + \frac{\cos^2\varphi\left(\cos^2\theta + \sin^2\theta\right)}{\rho} + \frac{\sin^2\varphi}{\rho} = \frac{\cos^2\theta - \sin^2\theta}{\rho} + \frac{\cos^2\varphi}{\rho} + \frac{\sin^2\varphi}{\rho} \\
&= \frac{1}{\rho} + \frac{1}{\rho} = \frac{2}{\rho}
\end{aligned}
\tag{11}
$$

$\frac{\partial}{\partial\theta}$:

$$
\begin{aligned}
&\frac{\partial\rho}{\partial x}\left[\frac{\partial}{\partial\rho}\left(\frac{d\theta}{dx}\right)\right] + \frac{d\theta}{dx}\left[\frac{\partial}{\partial\theta}\left(\frac{d\theta}{dx}\right)\right] + \frac{d\varphi}{dx}\left[\frac{\partial}{\partial\varphi}\left(\frac{d\theta}{dx}\right)\right] \\
&+ \frac{d\rho}{dy}\left[\frac{\partial}{\partial\rho}\left(\frac{d\theta}{dy}\right)\right] + \frac{d\theta}{dy}\left[\frac{\partial}{\partial\theta}\left(\frac{d\theta}{dy}\right)\right] + \frac{d\varphi}{dy}\left[\frac{\partial}{\partial\varphi}\left(\frac{d\theta}{dy}\right)\right] \\
&+ \frac{d\rho}{dz}\left[\frac{\partial}{\partial\rho}\left(\frac{d\theta}{dz}\right)\right] + \frac{d\theta}{dz}\left[\frac{\partial}{\partial\theta}\left(\frac{d\theta}{dz}\right)\right] + \frac{d\varphi}{dz}\left[\frac{\partial}{\partial\varphi}\left(\frac{d\theta}{dz}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sin\varphi\cos\theta\left[\frac{\partial}{\partial\rho}\left(\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\right)\right] + \frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\left[\frac{\partial}{\partial\theta}\left(\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\right)\right] + \frac{1}{\rho}\cos\varphi\cos\theta\left[\frac{\partial}{\partial\varphi}\left(\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\right)\right] \\
&+ \sin\varphi\sin\theta\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right)\right] + \frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right)\right] + \frac{1}{\rho}\cos\varphi\sin\theta\left[\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right)\right] \\
&+ \cos\varphi\left[\frac{\partial}{\partial\rho}(0)\right] + 0\left[\frac{\partial}{\partial\theta}(0)\right] + \frac{-1}{\rho}\sin\varphi\left[\frac{\partial}{\partial\varphi}(0)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\cos\theta\sin\theta}{\rho^2} + \frac{1}{\rho^2}\frac{\cos\theta\sin\theta}{\sin^2\varphi} + \frac{1}{\rho^2}\frac{\cos^2\varphi\cos\theta\sin\theta}{\sin^2\varphi} \\
&+ \frac{-1}{\rho^2}\cos\theta\sin\theta + \frac{-1}{\rho^2}\frac{\cos\theta\sin\theta}{\sin^2\varphi} + \frac{-1}{\rho^2}\frac{\cos^2\varphi\cos\theta\sin\theta}{\sin^2\varphi} \\
&= 0
\end{aligned}
\tag{12}
$$

$\frac{\partial}{\partial\varphi}$:

$$
\begin{aligned}
&\frac{d\rho}{dx}\left[\frac{\partial}{\partial\rho}\left(\frac{d\varphi}{dx}\right)\right] + \frac{d\theta}{dx}\left[\frac{\partial}{\partial\theta}\left(\frac{d\varphi}{dx}\right)\right] + \frac{d\varphi}{dx}\left[\frac{\partial}{\partial\varphi}\left(\frac{d\varphi}{dx}\right)\right] \\
&+ \frac{d\rho}{dy}\left[\frac{\partial}{\partial\rho}\left(\frac{d\varphi}{dy}\right)\right] + \frac{d\theta}{dy}\left[\frac{\partial}{\partial\theta}\left(\frac{d\varphi}{dy}\right)\right] + \frac{d\varphi}{dy}\left[\frac{\partial}{\partial\varphi}\left(\frac{d\varphi}{dy}\right)\right] \\
&+ \frac{d\rho}{dz}\left[\frac{\partial}{\partial\rho}\left(\frac{d\varphi}{dz}\right)\right] + \frac{d\theta}{dz}\left[\frac{\partial}{\partial\theta}\left(\frac{d\varphi}{dz}\right)\right] + \frac{d\varphi}{dz}\left[\frac{\partial}{\partial\varphi}\left(\frac{d\varphi}{dz}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sin\varphi\cos\theta\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\cos\varphi\cos\theta\right)\right] + \frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\cos\varphi\cos\theta\right)\right] + \frac{1}{\rho}\cos\varphi\cos\theta\left[\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\cos\varphi\cos\theta\right)\right] \\
&+ \sin\varphi\sin\theta\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\cos\varphi\sin\theta\right)\right] + \frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\cos\varphi\sin\theta\right)\right] + \frac{1}{\rho}\cos\varphi\sin\theta\left[\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\cos\varphi\sin\theta\right)\right] \\
&+ \cos\varphi\left[\frac{\partial}{\partial\rho}\left(\frac{-1}{\rho}\sin\varphi\right)\right] + 0\cdot\left[\frac{\partial}{\partial\theta}\left(\frac{-1}{\rho}\sin\varphi\right)\right] + \frac{-1}{\rho}\sin\varphi\left[\frac{\partial}{\partial\varphi}\left(\frac{-1}{\rho}\sin\varphi\right)\right]
\end{aligned}
$$

$$= \frac{-\sin\varphi\cos\varphi\cos^2\theta}{\rho^2} + \frac{1}{\rho^2}\frac{\cos\varphi\sin^2\theta}{\sin\varphi} + \frac{-1}{\rho^2}\cos\varphi\sin\varphi\cos^2\theta$$

$$+ \frac{-1}{\rho^2}\cos\varphi\sin\varphi\sin^2\theta + \frac{1}{\rho^2}\frac{\cos\varphi\cos^2\theta}{\sin\varphi} + \frac{-1}{\rho^2}\cos\varphi\sin\varphi\sin^2\theta$$

$$+ \frac{1}{\rho^2}\sin\varphi\cos\varphi + 0 + \frac{1}{\rho^2}\cos\varphi\sin\varphi \tag{13}$$

$$= \frac{-2\sin\varphi\cos\varphi}{\rho^2}\left(\cos^2\theta + \sin^2\theta\right) + \frac{1}{\rho^2}\frac{\cos\varphi}{\sin\varphi}\left(\sin^2\theta + \cos^2\theta\right)$$

$$+ \frac{2}{\rho^2}\sin\varphi\cos\varphi = \frac{1}{\rho^2}\frac{\cos\varphi}{\sin\varphi} = \frac{1}{\rho^2}\cot\varphi$$

$\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)$:

$$\frac{\partial\theta}{\partial x}\left[\frac{\partial\theta}{\partial x}\right] + \frac{\partial\theta}{\partial y}\left[\frac{\partial\theta}{\partial y}\right] + \frac{\partial\theta}{\partial z}\left[\frac{\partial\theta}{\partial z}\right]$$

$$= \frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\left[\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\right] + \frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\left[\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right] + 0[0]$$

$$= \frac{1}{\rho^2}\frac{\sin^2\theta}{\sin^2\varphi} + \frac{1}{\rho^2}\frac{1}{\sin^2\varphi} = \frac{1}{\rho^2}\frac{1}{\sin^2\varphi}\left(\sin^2\theta + \cos^2\theta\right) = \frac{1}{\rho^2}\frac{1}{\sin^2\varphi}$$

$\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right)$:

$$\frac{\partial\varphi}{\partial x}\left[\frac{\partial\varphi}{\partial x}\right] + \frac{\partial\varphi}{\partial y}\left[\frac{\partial\varphi}{\partial y}\right] + \frac{\partial\varphi}{\partial z}\left[\frac{\partial\varphi}{\partial z}\right]$$

$$= \frac{1}{\rho}\cos\varphi\cos\theta\left[\frac{1}{\rho}\cos\varphi\cos\theta\right] + \frac{1}{\rho}\cos\varphi\sin\theta\left[\frac{1}{\rho}\cos\varphi\sin\theta\right] + \frac{-1}{\rho}\sin\varphi\left[\frac{-1}{\rho}\sin\varphi\right]$$

$$= \frac{1}{\rho^2}\cos^2\varphi\cos^2\theta + \frac{1}{\rho^2}\cos^2\varphi\sin^2\theta + \frac{1}{\rho^2}\sin^2\varphi = \frac{1}{\rho^2}\cos^2\varphi\left(\cos^2\theta + \sin^2\theta\right) + \frac{1}{\rho^2}\sin^2\varphi$$

$$= \frac{1}{\rho^2}\cos^2\varphi + \frac{1}{\rho^2}\sin^2\varphi = \frac{1}{\rho^2}$$

$\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\theta}\right)$:

$$\frac{\partial\rho}{\partial x}\left[\frac{\partial\theta}{\partial x}\right] + \frac{\partial\rho}{\partial y}\left[\frac{\partial\theta}{\partial y}\right] + \frac{\partial\rho}{\partial z}\left[\frac{\partial\theta}{\partial z}\right]$$

$$= \sin\varphi\cos\theta\left[\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\right] + \sin\varphi\sin\theta\left[\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right] + \cos\varphi[0]$$

$$= \frac{-1}{\rho}\cos\theta\sin\theta + \frac{1}{\rho}\cos\theta\sin\theta + 0 = 0$$

$\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\varphi}\right)$:

$$\frac{\partial\rho}{\partial x}\left[\frac{\partial\varphi}{\partial x}\right]+\frac{\partial\rho}{\partial y}\left[\frac{\partial\varphi}{\partial y}\right]+\frac{\partial\rho}{\partial z}\left[\frac{\partial\varphi}{\partial z}\right]$$
$$=\sin\varphi\cos\theta\left[\frac{1}{\rho}\cos\varphi\cos\theta\right]+\sin\varphi\sin\theta\left[\frac{1}{\rho}\cos\varphi\sin\theta\right]+\cos\varphi\left[\frac{-1}{\rho}\sin\varphi\right]$$
$$=\frac{1}{\rho}\cos\varphi\sin\varphi\cos^2\theta+\frac{1}{\rho}\cos\varphi\sin\varphi\sin^2\theta+\frac{-1}{\rho}\cos\varphi\sin\varphi$$
$$=\frac{1}{\rho}\cos\varphi\sin\varphi\left(\cos^2\theta+\sin^2\theta\right)+\frac{-1}{\rho}\cos\varphi\sin\varphi=\frac{1}{\rho}\cos\varphi\sin\varphi+\frac{-1}{\rho}\cos\varphi\sin\varphi=0$$

$\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\rho}\right)$:

$$\frac{\partial\theta}{\partial x}\left[\frac{\partial\rho}{\partial x}\right]+\frac{\partial\theta}{\partial y}\left[\frac{\partial\rho}{\partial y}\right]+\frac{\partial\theta}{\partial z}\left[\frac{\partial\rho}{\partial z}\right]$$
$$=\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\left[\sin\varphi\cos\theta\right]+\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\left[\sin\varphi\sin\theta\right]+0\cdot\left[\cos\varphi\right]$$
$$=\frac{-1}{\rho}\cos\theta\sin\theta+\frac{1}{\rho}\cos\theta\sin\theta=0$$

$\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\varphi}\right)$:

$$\frac{\partial\theta}{\partial x}\left[\frac{\partial\varphi}{\partial x}\right]+\frac{\partial\theta}{\partial y}\left[\frac{\partial\varphi}{\partial y}\right]+\frac{\partial\theta}{\partial z}\left[\frac{\partial\varphi}{\partial z}\right]$$
$$=\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\left[\frac{1}{\rho}\cos\varphi\cos\theta\right]+\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\left[\frac{1}{\rho}\cos\varphi\sin\theta\right]+0\cdot\left[\frac{-1}{\rho}\sin\varphi\right]$$
$$=\frac{-1}{\rho^2}\frac{\cos\varphi\cos\theta\sin\theta}{\sin\varphi}+\frac{1}{\rho^2}\frac{\cos\varphi\sin\theta\cos\theta}{\sin\varphi}=0$$

$\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\rho}\right)$:

$$\frac{\partial\varphi}{\partial x}\left[\frac{\partial\rho}{\partial x}\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\partial\rho}{\partial y}\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\partial\rho}{\partial z}\right]$$
$$=\frac{1}{\rho}\cos\varphi\cos\theta\left[\sin\varphi\cos\theta\right]+\frac{1}{\rho}\cos\varphi\sin\theta\left[\sin\varphi\sin\theta\right]+\frac{-1}{\rho}\sin\varphi\left[\cos\varphi\right]$$
$$=\frac{1}{\rho}\sin\varphi\cos\varphi\cos^2\theta+\frac{1}{\rho}\cos\varphi\sin\varphi\sin^2\theta+\frac{-1}{\rho}\sin\varphi\cos\varphi$$
$$=\frac{1}{\rho}\sin\varphi\cos\varphi\left(\cos^2\theta+\sin^2\theta\right)+\frac{-1}{\rho}\sin\varphi\cos\varphi=\frac{1}{\rho}\sin\varphi\cos\varphi+\frac{-1}{\rho}\sin\varphi\cos\varphi=0$$

$\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta}\right)$:

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial x}\left[\frac{\partial\theta}{\partial x}\right]+\frac{\partial\varphi}{\partial y}\left[\frac{\partial\theta}{\partial y}\right]+\frac{\partial\varphi}{\partial z}\left[\frac{\partial\theta}{\partial z}\right]\\
&=\frac{1}{\rho}\cos\varphi\cos\theta\left[\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi}\right]+\frac{1}{\rho}\cos\varphi\sin\theta\left[\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right]+\frac{-1}{\rho}\sin\varphi[0]\\
&=\frac{-1}{\rho^2}\frac{\cos\varphi\sin\theta\cos\theta}{\sin\varphi}+\frac{1}{\rho^2}\frac{\cos\varphi\sin\theta\cos\theta}{\sin\varphi}=0
\end{aligned}
$$

非ゼロ係数項のみ抽出し，整理すると以下となる。

$$
\begin{aligned}
&Ss\frac{\partial s}{\partial t}=K_x\frac{\partial^2 s}{\partial x^2}+K_y\frac{\partial^2 s}{\partial y^2}+K_z\frac{\partial^2 s}{\partial z^2}=K\frac{\partial^2 s}{\partial x^2}+K\frac{\partial^2 s}{\partial y^2}+K\frac{\partial^2 s}{\partial z^2}\\
&=K\left[\frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\rho}\right)+\frac{2}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\cot\varphi\frac{\partial}{\partial\varphi}+\frac{1}{\rho^2}\frac{1}{\sin^2\varphi}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)\right]
\end{aligned}
$$

となり，極座標系における支配方程式が誘導できた。

【参考文献】

1） Kreyszig, E. : Advanced engineering mathematics, the seventh edition, John Wiley & Sons, Inc., pp.683-684, 1993.

追加誘導

以下の関係を誘導しておく。

$$\frac{\partial\rho}{\partial x}=\sin\varphi\cos\theta,\quad \frac{\partial\rho}{\partial y}=\sin\varphi\sin\theta,\quad \frac{\partial\rho}{\partial z}=\cos\varphi$$
$$\frac{\partial\varphi}{\partial x}=\frac{1}{\rho}\cos\varphi\cos\theta,\quad \frac{\partial\varphi}{\partial y}=\frac{1}{\rho}\cos\varphi\sin\theta,\quad \frac{\partial\varphi}{\partial z}=\frac{-1}{\rho}\sin\varphi \tag{23}$$
$$\frac{\partial\theta}{\partial x}=\frac{-1}{\rho}\frac{\sin\theta}{\sin\varphi},\quad \frac{\partial\theta}{\partial y}=\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi},\quad \frac{\partial\theta}{\partial z}=0$$

得られた定義は以下である。

$$(x,y,z)=(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi),\quad \rho^2=x^2+y^2+z^2$$
$$\rho=\sqrt{x^2+y^2+z^2},\quad \varphi=\cos^{-1}\left(\frac{z}{\rho}\right)=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right),\quad \theta=\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \tag{24}$$

ここで，原点 O と任意の座標点 P，点 P の x−y 平面投影点 Q を考え，

$\vec{OP}$ と z 軸の成す角度を φ，$\vec{OQ}$ と x 軸の成す角度を θ とする

$$\frac{d\rho}{dx}=\frac{d\left[\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\right]}{d}=\frac{1}{2}\cdot 2x\cdot\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}=\frac{\rho\sin\varphi\cos\theta}{\rho}=\sin\varphi\cos\theta \tag{25}$$

$$\frac{d\rho}{dy}=\frac{d\left[\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\right]}{d}=\frac{1}{2}\cdot 2y\cdot\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}=\frac{\rho\sin\varphi\sin\theta}{\rho}=\sin\varphi\sin\theta \tag{26}$$

$$\frac{d\rho}{dz}=\frac{d\left[\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\right]}{d}=\frac{1}{2}\cdot 2z\cdot\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}=\frac{\rho\cos\varphi}{\rho}=\cos\varphi \tag{27}$$

＜これ以降の誘導は次バージョンで掲載予定＞