直交座標系(x, y, z)から円筒座標系(r, θ, z)への変換

直交座標系で示された 3 次元基礎方程式は次式である。

$$K_x \frac{\partial^2 s}{\partial x^2} + K_y \frac{\partial^2 s}{\partial y^2} + K_z \frac{\partial^2 s}{\partial z^2} = Ss \frac{\partial s}{\partial t}$$

揚水問題では扱うことの多い円筒座標系で表記された以下の方程式となることを示すことが，この資料の目的である。

$$K_r \frac{\partial^2 s}{\partial r^2} + \frac{K_r}{r} \frac{\partial s}{\partial r} + K_z \frac{\partial^2 s}{\partial z^2} = Ss \frac{\partial s}{\partial t}$$

この座標変換は以下の手順で誘導できる。まず，ここでは K_x=K_y=K_r とし，これら水平方向は等方性として，鉛直方向は異方性を有すとする。

円筒座標系では座標 x, y, z はそれぞれ次式で表わされる。

$$(x, y, z) = (r\cos\theta, r\sin\theta, z)$$

ここで，r≥0，0≤θ≤2π である。つまり， $x = r\cos\theta$， $y = r\sin\theta$， $r = \sqrt{x^2 + y^2}$ 。

したがって，座標変換は x, y 項に対して行うため，これらの関係する項についてのみ解説する。

ここで，θ は x 軸と原点 O から点(x, y)に向かうベクトルの成す角である。

この資料では，直交座標 x, y 項による微分を，円筒座標 r, θ 項による微分に書き換えを行う。

ここで，$\partial/\partial x$ および$\partial/\partial y$ 項を得るために，以下の関係式を考える [1)]。

基本的事項の確認

合成関数の微分法：$h=f(x)$，$x=g(y)$としたとき，$h=f(g(y))$とすると，

$$\frac{dh}{dy} = \frac{dh}{dx}\frac{dx}{dy}$$

多変量関数の微分法(連鎖律：chain-rule)：$h=f(x,y)$，$x=x(r,\theta)$，$y=y(r,\theta)$とすると

$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial h}{\partial \theta} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial \theta}$$

上記の基本事項を活用し，さらに座標変換の定義から誘導できる以下の微分を用いると，(1)，(2)式が得られる。

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \tag{1}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \tag{2}$$

上記の 2 式を連立し，$\partial/\partial x$ および$\partial/\partial y$ に対して解く。

［式(1)×$r\cos\underline{\theta}$，式(2)×$\sin\theta$］

$$r\cos\theta\frac{\partial}{\partial r} = r\cos^2\theta\frac{\partial}{\partial x} + r\cos\theta\sin\theta\frac{\partial}{\partial y} \tag{3}$$

$$\sin\theta\frac{\partial}{\partial \theta} = -r\sin^2\theta\frac{\partial}{\partial x} + r\cos\theta\sin\theta\frac{\partial}{\partial y} \tag{4}$$

［式(3)－式(4)］

$$r\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial \theta} = r\cos^2\theta\frac{\partial}{\partial x} + r\sin^2\theta\frac{\partial}{\partial x} = r\left(\cos^2\theta + \sin^2\theta\right)\frac{\partial}{\partial x} = r\frac{\partial}{\partial x}$$

$$\therefore \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta} \tag{5}$$

さらに,

［式(1)×$r\sin\theta$，式(2)×$\cos\theta$］

$$r\sin\theta \frac{\partial}{\partial r} = r\cos\theta\sin\theta \frac{\partial}{\partial x} + r\sin^2\theta \frac{\partial}{\partial y} \tag{6}$$

$$\cos\theta \frac{\partial}{\partial \theta} = -r\cos\theta\sin\theta \frac{\partial}{\partial x} + r\cos^2\theta \frac{\partial}{\partial y} \tag{7}$$

［式(6)＋式(7)］

$$r\sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{\partial}{\partial \theta} = r\sin^2\theta \frac{\partial}{\partial y} + r\cos^2\theta \frac{\partial}{\partial y}$$

$$\therefore \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta} \tag{8}$$

＜誘導 1＞

よって，直交座標系での式にみられる$\partial^2/\partial x^2$および$\partial^2/\partial y^2$はそれぞれ以下となる。

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \left(\cos\theta\frac{\partial}{\partial r}\right)\left(\cos\theta\frac{\partial}{\partial r}\right) - \left(\cos\theta\frac{\partial}{\partial r}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - \left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r}\right) + \left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

各項ごとに整理してみる

$$\left(\cos\theta\frac{\partial}{\partial r}\right)\left(\cos\theta\frac{\partial}{\partial r}\right) = \cos^2\theta\frac{\partial^2}{\partial r^2} + \cos\theta\frac{\partial\cos\theta}{\partial r}\frac{\partial}{\partial r} = \cos^2\theta\frac{\partial^2}{\partial r^2}$$

$$-\left(\cos\theta\frac{\partial}{\partial r}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) = -\frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\right) - \cos\theta\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta} = -\frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\right) + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta}$$

$$-\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r}\right) = -\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right) - \frac{\sin\theta}{r}\frac{\partial\cos\theta}{\partial\theta}\frac{\partial}{\partial r} = -\frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right) + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

$$\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) = \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta} = \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \left(\sin\theta\frac{\partial}{\partial r}\right)\left(\sin\theta\frac{\partial}{\partial r}\right) + \left(\sin\theta\frac{\partial}{\partial r}\right)\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) + \left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r}\right) + \left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$

各項ごとに整理してみる

$$\left(\sin\theta\frac{\partial}{\partial r}\right)\left(\sin\theta\frac{\partial}{\partial r}\right) = \sin^2\theta\frac{\partial^2}{\partial r^2} + \sin\theta\frac{\partial}{\partial r}(\sin\theta)\frac{\partial}{\partial r} = \sin^2\theta\frac{\partial^2}{\partial r^2}$$

$$\left(\sin\theta\frac{\partial}{\partial r}\right)\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) = \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\right)\frac{\partial}{\partial\theta} = \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}$$

$$\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r}\right) = \frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial r} = \frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right) + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}$$

$$\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) = \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r}\right)\frac{\partial}{\partial\theta} = \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta}$$

$\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}$ を考えるので，上記の項の和をとり，以下ように整理できる

$$\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 s}{\partial r^2}+\left(\frac{\sin^2\theta+\cos^2\theta}{r}\right)\frac{\partial s}{\partial r}+\left(\frac{\sin\theta\cos\theta-\sin\theta\cos\theta}{r^2}+\frac{\sin\theta\cos\theta-\sin\theta\cos\theta}{r}\right)\frac{\partial s}{\partial\theta}$$
$$+\left(\frac{-\sin\theta\cos\theta}{r}+\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial}{\partial r}\left(\frac{\partial s}{\partial\theta}\right)+\left(\frac{-\sin\theta\cos\theta}{r}+\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial}{\partial\theta}\left(\frac{\partial s}{\partial r}\right)+\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 s}{\partial\theta^2}$$
$$=\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 s}{\partial r^2}+\left(\frac{\sin^2\theta+\cos^2\theta}{r}\right)\frac{\partial s}{\partial r}+\left(\frac{\cos^2\theta+\sin^2\theta}{r^2}\right)\frac{\partial^2 s}{\partial\theta^2}$$

$\cos^2\theta+\sin^2\theta=1$ より

$$\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}+\frac{1}{r^2}\frac{\partial^2 s}{\partial\theta^2}$$ となる。

さらに，軸対称問題と考えると，θ 方向には変動がないので，$\partial/\partial\theta=0$ となる

$$\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}$$

よって，まとめると以下となる。

$$K_x\frac{\partial^2 s}{\partial x^2}+K_y\frac{\partial^2 s}{\partial y^2}+K_z\frac{\partial^2 s}{\partial z^2}=K_r\frac{\partial^2 s}{\partial r^2}+\frac{K_r}{r}\frac{\partial s}{\partial r}+K_z\frac{\partial^2 s}{\partial z^2}$$

＜誘導 2＞

誘導 1 では, 直交座標 x, y による微分項を考えたが, ここでは円筒座標 r, θ による微分項を考える。まず, r についての二階微分を考える。

$$\frac{\partial^2}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial x}{\partial r}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\right)$$
$$=\frac{\partial x}{\partial r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}\right)+\frac{\partial^2 x}{\partial r^2}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial y}\right)+\frac{\partial^2 y}{\partial r^2}\frac{\partial}{\partial y}$$
$$=\frac{\partial x}{\partial r}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial r}\right)+\frac{\partial^2 x}{\partial r^2}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial r}\right)+\frac{\partial^2 y}{\partial r^2}\frac{\partial}{\partial y}$$
$$=\frac{\partial x}{\partial r}\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial r}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\right)+\frac{\partial^2 x}{\partial r^2}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial r}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\right)+\frac{\partial^2 y}{\partial r^2}\frac{\partial}{\partial y}$$

ここで,

$\frac{\partial x}{\partial r}=\cos\theta$， $\frac{\partial y}{\partial r}=\sin\theta$， $\frac{\partial^2 x}{\partial r^2}=0$， $\frac{\partial^2 y}{\partial r^2}=0$ を代入しておく。

$$\frac{\partial^2}{\partial r^2}=\cos\theta\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)+0\cdot\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)+0\cdot\frac{\partial}{\partial y}$$
$$=\cos^2\theta\frac{\partial^2}{\partial x^2}+\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+\sin^2\theta\frac{\partial^2}{\partial y^2}$$
$$=\cos^2\theta\frac{\partial^2}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+\sin^2\theta\frac{\partial^2}{\partial y^2}$$

同様に，θ についての二階微分は以下となる

$$\frac{\partial^2}{\partial\theta^2}=\frac{\partial x}{\partial\theta}\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial\theta}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\right)+\frac{\partial^2 x}{\partial\theta^2}\frac{\partial}{\partial x}+\frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial\theta}\frac{\partial}{\partial x}+\frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}\right)+\frac{\partial^2 y}{\partial\theta^2}\frac{\partial}{\partial y}$$

ここで,

$$\frac{\partial x}{\partial \theta}=-r\sin\theta,\quad \frac{\partial y}{\partial \theta}=r\cos\theta$$ および,

$$\frac{\partial^2 x}{\partial \theta^2}=\frac{\partial(-r\sin\theta)}{\partial\theta}=-r\cos\theta,\quad \frac{\partial^2 y}{\partial \theta^2}=\frac{\partial(r\cos\theta)}{\partial\theta}=-r\sin\theta$$

を代入しておく。

$$\frac{\partial^2}{\partial\theta^2}=(-r\sin\theta)\frac{\partial}{\partial x}\left((-r\sin\theta)\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right)-r\cos\theta\cdot\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\left(-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right)-r\sin\theta\cdot\frac{\partial}{\partial y}$$

$$=r^2\sin^2\theta\frac{\partial^2}{\partial x^2}-r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}-r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2}{\partial y^2}-r\left(\cos\theta\cdot\frac{\partial}{\partial x}+r\sin\theta\cdot\frac{\partial}{\partial y}\right)$$

ここで，前出の(1)式を再掲する。

$$\frac{\partial}{\partial r}=\frac{\partial x}{\partial r}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}$$

よって，以下となる

$$\frac{\partial^2}{\partial\theta^2}=r^2\sin^2\theta\frac{\partial^2}{\partial x^2}-2r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2}{\partial y^2}-r\left(\frac{\partial}{\partial r}\right)$$

以下を整理して，$\partial^2/\partial x\partial y$ 項を消去する。

$$\frac{\partial^2}{\partial r^2}=\cos^2\theta\frac{\partial^2}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+\sin^2\theta\frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2}{\partial\theta^2}=r^2\sin^2\theta\frac{\partial^2}{\partial x^2}-2r^2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2}{\partial y^2}-r\left(\frac{\partial}{\partial r}\right)\quad\times(1/r^2)$$

以下のように整理できた

$$\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}=(\cos^2\theta+\sin^2\theta)\frac{\partial^2}{\partial x^2}+(\cos^2\theta+\sin^2\theta)\frac{\partial^2}{\partial y^2}-\frac{1}{r}\left(\frac{\partial}{\partial r}\right)$$

$$\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{r}\left(\frac{\partial}{\partial r}\right)$$

これより

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\left(\frac{\partial}{\partial r}\right)$$

軸対称問題と捉えると θ に関する微分項は 0 である。

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\left(\frac{\partial}{\partial r}\right)$$

解説

円筒座標系では座標 x, y, z はそれぞれ次式で表わされる.

$$(x,y,z)=(r\cos\theta, r\sin\theta, z)$$

ここで，$r\geq 0$，$0\leq\theta\leq 2\pi$ である

$$r=\sqrt{x^2+y^2},\quad \cos\theta=\frac{x}{r}\quad \sin\theta=\frac{y}{r}$$

$$\theta=\sin^{-1}\left(\frac{y}{r}\right)=\cos^{-1}\left(\frac{x}{r}\right)=\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x}=\frac{\partial(x^2+y^2)^{1/2}}{\partial x}=2x\cdot\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta$$

$$\frac{\partial r}{\partial y}=\frac{\partial\left(x^2+y^2\right)^{1/2}}{\partial y}=2y\cdot\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}=\frac{y}{r}=\sin\theta$$

$$\boxed{\frac{\partial\theta}{\partial x}=\frac{\partial\left(\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)}{\partial x}}$$

ここで，$t=\frac{y}{\sqrt{x^2+y^2}}$ とすると

$$\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial t}\left(\sin^{-1}t\right)\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right)=\frac{1}{\sqrt{1-t^2}}y\frac{\partial}{\partial x}\left(x^2+y^2\right)^{\frac{-1}{2}}$$

$$=\frac{1}{\sqrt{1-\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}}y\cdot 2x\cdot\frac{-1}{2}\left(x^2+y^2\right)^{\frac{-3}{2}}=\frac{-1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}y\cdot x\left(r^2\right)^{\frac{-3}{2}}=\frac{-1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}y\cdot x\cdot r^{-3}$$

$$=\frac{-1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\frac{y}{r}\cdot\frac{x}{r}\cdot\frac{1}{r}=\frac{-1}{\sqrt{1-\sin^2\theta}}\sin\theta\cdot\cos\theta\cdot\frac{1}{r}=\frac{-1}{\sqrt{\cos^2\theta}}\sin\theta\cdot\cos\theta\cdot\frac{1}{r}=\frac{-\sin\theta}{r}$$

$$\boxed{\frac{\partial\theta}{\partial y}=\frac{\partial\left(\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)}{\partial y}}$$

ここで，$t=\frac{x}{\sqrt{x^2+y^2}}$ とすると，

$$\frac{\partial\theta}{\partial y}=\frac{\partial}{\partial t}\left(\cos^{-1}t\right)\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)=\frac{-1}{\sqrt{1-t^2}}x\frac{\partial}{\partial y}\left(x^2+y^2\right)^{\frac{-1}{2}}$$

$$=\frac{-1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}x\cdot 2y\cdot\frac{-1}{2}\left(x^2+y^2\right)^{\frac{-3}{2}}=\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}x\cdot y\left(r^2\right)^{\frac{-3}{2}}=\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}x\cdot y\cdot r^{-3}$$

$$=\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}\frac{x}{r}\cdot\frac{y}{r}\cdot\frac{1}{r}=\frac{1}{\sqrt{1-\cos^2\theta}}\cos\theta\cdot\sin\theta\cdot\frac{1}{r}=\frac{1}{\sqrt{\sin^2\theta}}\sin\theta\cdot\cos\theta\cdot\frac{1}{r}=\frac{\cos\theta}{r}$$

以上のように計算できるが，先に求めた(5)，(8)式から以下の関係が誘導できる。

$$\frac{\partial r}{\partial x}=\cos\theta\frac{\partial r}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial r}{\partial\theta}=\cos\theta\frac{\partial r}{\partial r}$$

$$\frac{\partial\theta}{\partial x}=\cos\theta\frac{\partial\theta}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial\theta}{\partial\theta}=-\frac{1}{r}\sin\theta$$

$$\frac{\partial r}{\partial y}=\sin\theta\frac{\partial r}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial r}{\partial\theta}=\sin\theta\frac{\partial r}{\partial r}$$

$$\frac{\partial\theta}{\partial y}=\sin\theta\frac{\partial\theta}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial\theta}{\partial\theta}=\frac{1}{r}\cos\theta$$

【参考文献】

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